The Seifert-van Kampen theorem in Category

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Abstract

The Seifert-van Kampen theorem states the relationship between the fundermental groupoid of space X and colimts of the fundermental groupoid of path-connected open subsets which generates X. We shall conclude that functors of fundermental group category preserve colimits(pushouts) . More generally, functors of fundermental groupoid category preserve colimits(pushouts) .

Keywords: The Seifert-van Kampen Theorem, colimit, groupoid

1 Introduction

In topology, Seifert-van Kampen theorem, discovered independently by H. Seifert and E. Van Kampen, is an important theorem to caculate the fundermental group of topological sapce. However,we can narrate the theorem in category language. Basic definitions of categories, functors, natural transformations, colimits, co-products , pushouts are in the Appendix.

To begin with, we take a brief introduction of the simple version of the Seifert-van Kampen Theorem without proof, in that we allow X to be the union of two path-connected open subsets, and enumerate an example to better understand it.

In the next, we introduce the definition of groupoid and complete the proof theorem in terms of finite categories.

Furthermore, there is a more general version of the theorem, which allows X to be the union of infinite number of path-connected open subsets. Then we use the theorem to determine the structure of the fundamental groups of compact surfaces.

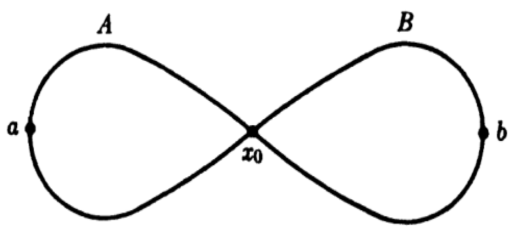
2 The Seifert-van Kampen theorem in topological space

Theorem 1. *Let X be a topological space covered by two path-connected open sets U_1 and U_2, x is a base point in both U_1 and U_2, then we have*

$$\pi_1(X, x_0) \cong \pi_1(U_1, x_0) * \pi_1(U_2, x_0)/G$$

where G is a normal subgroup generated by all elements of the form $(i_1)_*(\alpha)(i_2)_*(\alpha)^{-1}$, α is in $\pi_1(U_1 \cap U_2, x_0)$ and $(i_1)_*, (i_2)_*$ are induced by inclusion maps.

Let X be a space that $X = A \cup B, A \cap B = \{x_0\}$,and A,B each homeomorphic to a circle S^1.



Considering that A,B are not open sets in X, we can't use Seifert-van Kampen Theorem directly. Choose points $a \in A$ and $b \in B$ such that $a, b \neq x_0$. Let open set $U = X - \{b\}$ and open set $V = X - \{a\}$,we can get that $U \cap V = X - \{a, b\} \simeq \{x_0\}$(in that $\{x_0\}$ is a deformation retract of$U \cap V$), So $U \cup V$ is simply connected. According to the theorem above, thus $\pi(X) = \pi(U) * \pi(V)$. Because $V \simeq B$(in that B is a deformation retract of U),$U \simeq A,\ A, B \cong S^1$, then $\pi(X) = \pi(A) * \pi(B) = F(< a_1 >, < a_2 >)$,where $a_1(a_2)$is a loop with a base point x_0 goes once around $A(B)$.

3 The Seifert-van Kampen Theorem

Now,we state a general version of the Seifert-Van Kampen theorem. The generalization consists in allowing a covering of the space X by finite numbers of open sets in category language.The basic definitions of categories related to the theorem is in the Appendix, you can skim through it before going through the following proofs.

Theorem 2. *Let X be a path connected space, $\mathscr{U} = \{U_i | i = 1, ..., n\}$ is a cover of X by path connected open subsets such that the intersection of subsets in U is again in $\mathscr{U}$. Regard $\mathscr{U}$ as a category whose objects are $U_1, ..., U_n$ and morphisms are the inclusion of subsets. Considering the diagram of groups $\pi_1|_{\mathscr{U}} : \mathscr{U} \to \mathscr{G}$ dedined by $\pi|_{\mathscr{U}}(U_i) = \pi_1(U_i, x)$, where $\mathscr{G}$ is the category of Groups and homomorphisms. Then the group $\pi_1(X, x)$ is the colimit of the diagram.i.e.*

$$\pi_1(X, x) \cong colim \pi_1(U_i, x) \qquad i = 1, ..., n$$

Before proving the theorem, we can take the type $X = U_1 \cup U_2$ to help us understand the theorem explictly, refering to *in Derived Algebraic Geometry VI: E[k]-Algebras* .

Let X be a topological space covered by a pair of open sets U and V, such that U, V, and $U \cap V$ are path-connected. The Seifert-van Kampen theorem asserts that, for any choice of base point $x \in U \cap V$, the diagram of groups

$$\begin{tikzcd} \pi_1(U \cap V, x) \arrow[r] \arrow[d] & \pi_1(U, x) \arrow[d] \\ \pi_1(V, x) \arrow[r] & \pi_1(X, x) \end{tikzcd}$$

is a pushout square. In this section, we will prove a generalization of the Seifert-van Kampen theorem, which describes the entire weak homotopy type of X in terms of any sufficiently nice covering of X by open sets:

To prove the theorem, we introduce several definitions.

Definition 1 (Fundermental Groupoid)**.** *Let X be a space, define the category $\Pi(X)$ whose objects are points of X and morphisms are path homotopy classes $[f] : x \to y$ in X, $\Pi(X)$ is called the fundermental groupoid of X.*

Definition 2 (A skeleton of Category)**.** *Let $\mathscr{C}$ be a category ,a skeleton $SK\mathscr{C}$ is a full subcategory that the morphisms between two objects of $SK\mathscr{C}$ are all of the morphisms between these objects in $\mathscr{C}$.*

Lemma 1. *Regarding $\pi_1(X, x)$ as a category with a single object x, then it is a skeleton of $\pi_1(X, x)$.*

Lemma 2. *The inclusion functor $J : SK\mathscr{C} \to \mathscr{C}$ is an equvalence of categories.*

proposition 1. Let X be a path connected space,$\mathscr{U} = \{U\}$ is a cover of X by path connected open subsets such that the intersection of finitely many subsets in U are again in U. Regard $\mathscr{U}$ as a category whose objects are those U and morphisms are the inclusion of subsets. Considering the functor Π, resticted to the spaces and maps in $\mathscr{U}$, it gives a diagram $\Pi|_{\mathscr{U}} : \mathscr{U} \to \mathscr{G}$ of groupoid. The groupoid $\Pi(X)$ is the colimit of the diagram. i.e.

$$\Pi(X) \cong colim_{U \in \mathscr{U}} \Pi(U) \qquad i = 1, ..., n$$

Proof. *According to Lemma 1 and Lemma 2, we have that the inclusion functor*$\pi_1(X, x) \to \Pi(X)$ *is an equivalence of categories. Next,we verify the universial property in the category of groups. That's to say, for a group G and a mapp* $: \pi_1|\mathscr{U} \to G$ *of* $\mathscr{U}$*-shaped diagrams of groups, we shall show that there is a unique homomoephism*$\hat{p} : \pi_1(X, x) \to G$ *that restricts to* η_{U_i} *on* $\pi_1(U_i, x)$.

As the inclusion $J : \pi_1(X, x) \to \Pi(X)$ *is an equivalence,we denote its inverse equivalence* $F : \Pi(X) \to \pi_1(X, x)$*,determined by a choice of path homotopy classes* $f : x \to y for y \in X$. *Given that the cover is finite and closed under finite intersections, we can choose good paths to make the path* $f : x \to y$*lies entirely in* U_i *for any* y.

Then, the choosen paths determine compatible inverse equivalences $F_{U_i} : \Pi(U_i) \to \pi_1(U_i, x)$ *to the inclusions* $J_{U_i} : \pi_1(U_i, x) \to \Pi(U_i)$. *Thus the functors*

$$\Pi(U_i) \to \pi_1(U_i, x) \to \mathscr{G}$$

gives an $\mathscr{U}$*-shaped diagram of groupoid* $\Pi(\mathscr{U}) \to \mathscr{G}$. *So we verify the universial property in the category of groups.*

4 Generalization of The Seifert-van Kampen Theorem

Theorem 3. *Let X be a path connected space,*$\mathscr{U} = \{U\}$ *is a cover of X by path connected open subsets such that the intersection of arbitrary many subsets in* $\mathscr{U}$ *are again in* $\mathscr{U}$. *Regard U as a category whose objects are those U and morphisms are the inclusion of subsets. Considering the funtor* $\pi_1(-, x)$, *resticted to the spacces and maps in* $\mathscr{U}$, *it gives a diag* $\pi_1|_{\mathscr{U}} : \mathscr{U} \to \mathscr{G}$ *of fundermental group. The fundermental group* $\pi_1(X, x)$ *is the colimit of* *the diagram. i.e.*

$$\pi_1(X,x) \cong colim_{U\in\mathscr{U}}\pi_1(U,x) \qquad i=1,...,n$$

Proof. *In the following, we just have a sketch of the proof. Let $\mathscr{F}$ be the set of those finite susbets of the cover $\mathscr{U}$ that are closed under finite intersection. For $\mathscr{K} \in mathscrF$,let $\mathscr{U}_{\mathscr{K}}$ be the union of the U in $\mathscr{K}$. Then$\mathscr{K}$ is a finite cover of $\mathscr{U}_{\mathscr{K}}$.According to theorem above, we have*

$$colim_{\mathscr{U}\in\mathscr{K}}\pi_1(U,x) \cong \pi_1(\mathscr{U}_{\mathscr{K}},x)$$

Regarding $\mathscr{F}$ as a category with a morphism $\mathscr{K} \to \mathscr{M}$ whenever $\mathscr{U}_{\mathscr{K}} \subset \mathscr{U}_{\mathscr{M}}$. If we verify : ***(1)*** *$colim_{\mathscr{U}\in\mathscr{K}}\pi_1(\mathscr{U}_{\mathscr{K}},x) \cong \pi_1(X,x)$* ***(2)*** *$colim_{U\in\mathscr{U}}\pi_1(U,x) \cong colim_{\mathscr{U}\in\mathscr{K}}\pi_1(\mathscr{U}_{\mathscr{K}},x)$, then we can finish the proof.*

The proof of (1): *By the usual subdivision argument, we can prove that $\pi_1(X,x)$ together with the homomorphism $\pi_1(U_{\mathscr{K}},x) \to \pi_1(X,x)$ has the universial property that cgaracterizes the claimed colimit, then*

$$colim_{\mathscr{U}\in\mathscr{K}}\pi_1(U_{\mathscr{K}},x) \cong \pi_1(X,x)$$

The proof of (2) *Firstly, substituing the colimit on the right of (2), we get*

$$colim_{\mathscr{K}\in\mathscr{F}}\pi_1(\mathscr{U}_{\mathscr{K}},x) \cong colim_{K\in\mathscr{F}}colim_{U\in\mathscr{K}}\pi_1(U,x) \cong colim_{(\mathscr{U},\mathscr{K})\in(\mathscr{U},\mathscr{F})}\pi_1(U,x)$$

here $(\mathscr{U},\mathscr{F})$ is an index category. Also, according to the comparision of universial properties, then we have an induced map of colimits:

$$colim_{U\in\mathscr{U}}\pi_1(U,x) \to colim_{(\mathscr{U},\mathscr{K})\in(\mathscr{U},\mathscr{F})}\pi_1(U,x)$$

Again, from construct projection th the first coordinate, we can get a functor

$$R : (\mathscr{U},\mathscr{F}) \to \mathscr{U}$$

Then the composition of $\mathscr{G}$ and $\pi_1(-,x) : \mathscr{U} \to \mathscr{G}$ defines a colimit. Then we can get another map of colimits:

$$colim_{(U,\mathscr{K})\in(\mathscr{U},\mathscr{F})}\pi_1(U,x) \to colim_{U\in\mathscr{U}}\pi_1(U,x)$$

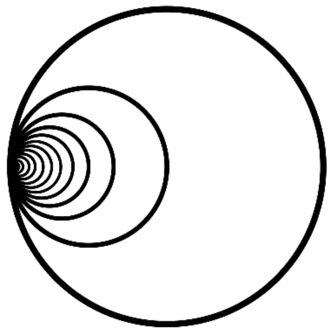
Then, we proved that $colim_{U\in\mathscr{U}}\pi_1(U,x) \cong colim_{(U,\mathscr{K})\in(\mathscr{U},\mathscr{F})}\pi_1(U,x)$.
(Referring to A Concise Course in Algebraic Topology,Page 19)

Example 4.1. According to the classification theorem of compact surface, any compact surface is homeomorphic to a sphere, or to a connected sum of tori $T^2 = S^1 \times S^1$, or to a connected sum of projective planes $RP^2 = S^2/Z^2$. We know that $\pi_1(T^2) \cong Z \times Z; \pi_1(RP^2) \cong Z_2; \pi_1(S^2) \cong 0$. Based on them and the van Kampen theorem, we can compute the fundamental group of various compact surfaces .

We have the following conclusion in terms of fundermental groups of compact surface:

$$\pi_1(mP^2) = F(\alpha_1, ..., \alpha_m)/[\alpha_1^2, ..., \alpha_m^2]$$
$$\pi_1(nT^2) = F(\alpha_1, \beta_1, ..., \beta_n)/[\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}...\alpha_n\beta_n\alpha_n^{-1}\beta_n^{-1}]$$

Example 4.2. Also, we can discuss the fundermental group of Shrinking wedge of circles.



Let X be the union of circles C_n of radius $1/n$ and center $(1/n, 0)$ for positive integer n. Considering the retraction $R_n : X \to Cn$,it exhibits all circles except C_n to the origion, then R_n induces a subjective map $h_n : \pi_1(X) \to \pi_1(C_n)$. We take the origion as a basepoint, the products of h_n is a homomorphism $h : \pi_1(X) \to \prod_{n=1}^{\infty} Z$. We should prove h is a subjective map. For any sequence K_n, we can construct a loop $f : I \to X$ which goes around C_n for K_n times restricted in the time interval $[1-1/n, 1/(n+1)]$. So $\pi_1(X)$ has more elments than $\prod_{n=1}^{\infty} Z$. This infinite composition of loops is continuous at each time less than 2.

Given that the fundermental group of countable wedge sum of circles is countably generated, then it is also countable. However, the fundermental group of shrinking wedge of circles has more elments than $\prod_{n=1}^{\infty} Z$, and $\prod_{n=1}^{\infty} Z$ is uncountable, thus the fundermental group of shrinking wedge of circles $\pi_(X)$ is uncountable, and it is more complicated than $\prod_{n=1}^{\infty} Z$.